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More on the Bruhat order for (0, 1)-matrices

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Dedicated to Mirek Fiedler on the occasion of his 80th birthday

Abstract

Let $\mathscr{A}(R, S)$ denote the class of all (0, 1)-matrices with row sum vector R and column sum vector S. Continuing an earlier investigation of the Bruhat order and secondary Bruhat order (both of which extend the classical Bruhat order on permutations of $\{1, 2, \ldots, n\}$) on $\mathscr{A}(R, S)$, we provide a counterexample to a conjecture of Brualdi and Hwang which shows that these two orders are not in general the same. We characterize the cover relation for the secondary Bruhat order. We also study in more detail certain classes $\mathscr{A}(R, S)$ where $R = S = (k, k, \ldots, k)$, a constant vector. We show that for $k = 2$ the Bruhat order and secondary Bruhat order are the same, but this is not always so when $k = 3$.

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1. Introduction

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be positive integral vectors. Then $\mathscr{A}(R, S)$ denotes the class of all m by n (0, 1)-matrices with row sum vector R and column sum vector S. Combinatorial properties of the class $\mathscr{A}(R, S)$ have been extensively investigated (see [6,2] and the forthcoming book [4]). In particular, necessary and sufficient conditions are known for $\mathscr{A}(R, S)$ to be nonempty; namely, assuming without loss of generality that R and S are nonincreasing vectors,

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$S \preceq R^*$ where R^* is the conjugate of R and $\preceq$ is the majorization (also called dominance) order. One important case in which nonemptiness is assured occurs when k is an integer with $0 \leqslant k \leqslant n$ and $R = S = (k, k, \ldots, k)$ are constant vectors with each component equal to k. In this case we write $\mathscr{A}(n, k)$ instead of $\mathscr{A}(R, S)$.

A basic result for the class $\mathscr{A}(R, S)$ is that given matrices A and B in $\mathscr{A}(R, S)$, A can be transformed into B by a sequence of *interchanges*

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

that replace a submatrix of order 2 equal to L_2 by I_2, or the other way around. Note that an interchange always takes a matrix in a class $\mathscr{A}(R, S)$ to another matrix in the same class.

A permutation τ of $\{1, 2, \ldots, n\}$ can be identified with the permutation matrix $P = [p_{ij}]$ of order n where $p_{ij} = 1$ if $j = \tau(i)$ and $p_{ij} = 0$, otherwise. In this way the set S_n of permutations of $\{1, 2, \ldots, n\}$ is identified with the class $\mathscr{A}(n, 1)$. In the class $\mathscr{A}(n, 1)$, an interchange $L_2 \to I_2$ that replaces an L_2-submatrix with an I_2-submatrix corresponds to a transposition

$$(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) \to (a_1, \ldots, a_j, \ldots, a_i, \ldots, a_n), \quad \text{where } a_i > a_j, \tag{1}$$

that decreases the number of inversions.

The *Bruhat order* on S_n is defined as follows: Let π and τ be permutations in S_n. Then $\pi \preceq_B \tau$ provided that π can be obtained from τ by a sequence of inversion-reducing transpositions (1). There is an equivalent way (see e.g. [1] and the discussion in [3]) to define the Bruhat order on S_n which, for purposes of generalization, we replace with $\mathscr{A}(n, 1)$. First, for an m by n matrix $A = [a_{ij}]$, let $\Sigma_A = [\sigma_{ij}(A)]$ be the m by n matrix where

$$\sigma_{ij}(A) = \sum_{k=1}^{i} \sum_{l=1}^{j} a_{kl} \quad (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n).$$

Then, see e.g. [5], if P and Q are the permutation matrices corresponding to π and τ, $\pi \preceq_B \tau$ if and only if

$$\Sigma_P \geqslant \Sigma_Q \text{ (entrywise order).}$$

In [3] a Bruhat (partial) order on a nonempty class $\mathscr{A}(R, S)$ was defined using this equivalent characterization of the Bruhat order on S_n. If A_1 and A_2 are in $\mathscr{A}(R, S)$, then A_1 *precedes* A_2 *in the Bruhat order*, written $A_1 \preceq_B A_2$, provided that in the entrywise order $\Sigma_{A_1} \geqslant \Sigma_{A_2}$. Implicit in [3], but not formally defined, is another partial order that we call the *secondary Bruhat* (*partial*) *order*. If A_1 and A_2 are in $\mathscr{A}(R, S)$, then A_1 *precedes* A_2 *in the secondary Bruhat order*, written $A_1 \preceq_{\hat{B}} A_2$, provided that A_2 can be transformed into A_1 by a sequence of $L_2 \to I_2$ interchanges. It is immediate that $A_1 \preceq_{\hat{B}} A_2$ implies that $A_1 \preceq_B A_2$, that is, the *Bruhat order is a refinement of the secondary Bruhat order*. It follows from our discussion above that the Bruhat order agrees with the secondary Bruhat order on $\mathscr{A}(n, 1)$. As usual, we write $A_1 \prec_B A_2$ to mean $A_1 \preceq A_2$ but $A_1 \neq A_2$; $A_1 \prec_{\hat{B}} A_2$ is defined in a similar way.

We now briefly discuss the remaining contents of [3]. An algorithm was given to construct a minimal matrix in the Bruhat order on $\mathscr{A}(R, S)$; this algorithm simplified for the classes $\mathscr{A}(n, k)$. Since the Bruhat order is a refinement of the secondary Bruhat order, it follows that the minimal matrices in the Bruhat order are also minimal in the secondary Bruhat order. As noted in [3] an algorithm that produces a minimal matrix in the Bruhat order can be used to produce a maximal matrix in the Bruhat order. In the Bruhat order on $\mathscr{A}(n, 1)$, the unique minimal matrix is the identity matrix I_n and the unique maximal matrix is the permutation matrix D_n with 1's in positions $(1, n), (2, n-1), \ldots, (n, 1)$. In general there may be many minimal and maximal matrices. The minimal matrices in the Bruhat order on $\mathscr{A}(n, 2)$ and $\mathscr{A}(n, 3)$ were characterized in [3]; they were direct sums of two types of matrices in the case of $\mathscr{A}(n, 2)$ and five types in the case of $\mathscr{A}(n, 3)$. Evidence was presented which suggested that, in general, there might not be any nice characterization of the minimal matrices in $\mathscr{A}(n, k)$.

In [3] it was conjectured (at least in the case that R and S are nonincreasing) that a matrix in $\mathscr{A}(R, S)$ is minimal in the Bruhat order if it has no submatrix of order 2 equal to L_2, that is, if it is minimal in the secondary Bruhat order (the converse is clear). Implicit was the conjecture that the Bruhat order and secondary Bruhat order are the same on $\mathscr{A}(R, S)$.

The organization of the remainder of this paper is the following. In Section 2 we construct counterexamples to the conjecture thereby proving that, in general, the Bruhat order is a proper refinement of the secondary Bruhat order. In Section 3 we characterize the cover relation for the secondary Bruhat order in a way that generalizes the cover relation for the classical Bruhat order on S_n. At this time we are unable to characterize the cover relation for the Bruhat order on general $\mathscr{A}(R, S)$. In Section 4 we restrict our attention to $\mathscr{A}(n, 2)$ and show, perhaps surprisingly, that the Bruhat order and secondary Bruhat order agree in this case. In Section 5 we consider the general classes $\mathscr{A}(n, k)$ and give an example to show that the Bruhat and secondary Bruhat orders are not the same on $\mathscr{A}(6, 3)$. We also characterize those classes $\mathscr{A}(n, k)$ that contain a unique minimal matrix. In the concluding section we mention some additional observations and present some open problems.

2. Counterexample to the conjecture

Our understanding of the elements of $\mathscr{A}(R, S)$ that are minimal with respect to the secondary Bruhat order is aided by the following:

Lemma 2.1. *Suppose $A \in \mathscr{A}(R, S)$ where R and S are nonincreasing integral vectors. If A is minimal in the secondary Bruhat order, then every leading submatrix of A has nonincreasing row and column sum vectors.*

Proof. Let $A = [a_{ij}]$ be minimal with respect to the secondary Bruhat order on $\mathscr{A}(R, S)$. Assume to the contrary that the i by j leading submatrix $M = A[\{1, 2, \ldots, i\}, \{1, 2, \ldots, j\}]$ of A does not have nonincreasing row and column sum vectors.

Suppose first that the column sum vector of M is not nonincreasing. Then there are integers k and l with $1 \leqslant k < l \leqslant j$ such that the sum of the entries of column k of M is less than that of column l. In particular, this implies that there is some integer p with $1 \leqslant p \leqslant i$ such that $a_{pk} = 0$ and $a_{pl} = 1$. Since the sum of the entries of column k of A is at least the sum of the entries of column l of A, it follows that there must be some integer q with $i < q \leqslant m$ such that $a_{qk} = 1$ and $a_{ql} = 0$. Since $p < q$, this implies that A has a submatrix equal to L_2, a contradiction to the assumption that A is minimal with respect to the secondary Bruhat order.

A similar argument works if the row sum vector of M is not nonincreasing. $\square$

Any element that is minimal with respect to the Bruhat order must also be minimal with respect to the secondary Bruhat order. A counterexample to the stated conjecture certainly exists if there exist two different matrices A and A' in a class $\mathscr{A}(R, S)$ that are both minimal with respect to the secondary Bruhat order, but that satisfy $A \prec_B A'$.

Suppose we have such matrices A and A'. It follows immediately from Lemma 2.1 that we may assume without loss in generality that A and A' differ in both their final row and in their final column. It also follows that A and A' *cannot* differ in their initial row or initial column. Let D_A be the smallest submatrix of A containing all the entries in which A and A' differ, and define $D_{A'}$ similarly. Then D_A and $D_{A'}$ have the same row sum vectors and the same column sum vectors (but these row and column sum vectors are not necessarily nonincreasing). Both D_A and $D_{A'}$ are minimal with respect to the secondary Bruhat order and $D_A \prec_B D_{A'}$. In our search for a counterexample then, we begin by finding such matrices. In particular, we can take

$$D_A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D_{A'} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Neither D_A nor $D_{A'}$ has a submatrix equal to L_2 and hence both are minimal in the secondary Bruhat order, even though $D_A \prec_B D_{A'}$. These matrices do not satisfy the monotonicity requirement of the conjecture, but, as detailed above, we can use them to find other matrices A and A' that do. In particular, we define matrices X and Y by

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and then let

$$A = \begin{bmatrix} J_7 & Y \\ X & D_A \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} J_7 & Y \\ X & D_{A'} \end{bmatrix}.$$

Here J_7 is the matrix of all 1's of order 7. The matrices A and A' form a counterexample to the conjecture as originally formulated in [3].

In some sense, the counterexample above is provided by "padding" a counterexample to the nonmonotone case to create one that satisfies the monotonicity condition. The question still remains as to whether the analogous conjecture holds in the more restrictive case of $\mathscr{A}(n, k)$. We will address this question in a later section.

3. The cover relation for the secondary Bruhat order

Let $(X, \leqslant)$ be a finite partially ordered set. For $a, b \in X$, b *covers* a (a is *covered* by b) provided $a < b$ and there does not exist an element $c \in X$ with $a < c < b$. The cover relation determines the partial order uniquely. In this section, we characterize the cover relation for the secondary Bruhat order on a class $\mathscr{A}(R, S)$. We begin by reviewing the cover relation for the classical Bruhat order on S_n.

The partially ordered set $(S_n, \preceq_B)$ is graded by the number of inversions. Let $\pi = (a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n)$ be a permutation in S_n with $i < j$ and $a_i > a_j$. Then the permutation $\tau = (a_1, \ldots, a_j, \ldots, a_i, \ldots, a_n)$ obtained from π by the transposition that interchanges a_i and a_j is covered by π if and only if each a_k with $i < k < j$ satisfies $a_k < a_j$ or $a_k > a_i$. In terms of matrices in $\mathscr{A}(n, 1)$, this means the following. Let P and Q be the permutation matrices in

$\mathscr{A}(n,1)$ corresponding to π and τ, respectively. Then P covers Q in the (secondary) Bruhat order if and only if the submatrix $P[\{i,i+1,\ldots,j\},\{a_j,a_j+1,\ldots,a_i\}]$ of P determined by the consecutive rows $i,i+1,\ldots,j$ and the consecutive columns $a_j,a_j+1,\ldots,a_i$ has the form

$$P[\{i,i+1,\ldots,j\},\{a_j,a_j+1,\ldots,a_i\}]=\begin{bmatrix}0&0&\cdots&0&1\\0&&&&0\\\vdots&&O&&\vdots\\0&&&&0\\1&0&\cdots&0&0\end{bmatrix},\tag{2}$$

that is, has exactly two 1's and these occur in the upper right and lower left corners. The corresponding submatrix of Q is that obtained by performing the $L_2\to I_2$ interchange on the matrix (2).

The cover relation for the secondary Bruhat order on a class $\mathscr{A}(R,S)$ is a little more complicated. Before stating it, we consider an example.

Let $R=S=(2,2,2,2)$. The matrices

$$A_1=\begin{bmatrix}1&1&0&0\\0&1&0&1\\1&0&1&0\\0&0&1&1\end{bmatrix}\quad\text{and}\quad A_2=\begin{bmatrix}0&1&0&1\\0&1&0&1\\1&0&1&0\\1&0&1&0\end{bmatrix}$$

belong to $\mathscr{A}(4,2)$. Although A_1 results from A_2 by one $L_2\to I_2$ interchange, we show that A_2 does not cover A_1 in the either the secondary Bruhat order or Bruhat order on $\mathscr{A}(4,2)$. In fact, let

$$C=\begin{bmatrix}0&1&1&0\\0&1&0&1\\1&0&1&0\\1&0&0&1\end{bmatrix}.$$

Then C results from A_2 by one $L_2\to I_2$ interchange, and A_1 results from C by another $L_2\to I_2$ interchange so that $A_1\prec_{\hat{B}}C\prec_{\hat{B}}A_2$.

Theorem 3.1. *Let $A=[a_{ij}]$ be a matrix in $\mathscr{A}(R,S)$ where $A[\{i,j\},\{k,l\}]=L_2$. Let $A'=[a'_{ij}]$ be the matrix obtained from A by the $L_2\to I_2$ interchange that replaces $A[\{i,j\},\{k,l\}]=L_2$ with I_2. Then A covers A' in the secondary Bruhat order on $\mathscr{A}(R,S)$ if and only if*

(i) $a_{pk}=a_{pl}$ $(i<p<j)$,
(ii) $a_{iq}=a_{jq}$ $(k<q<l)$,
(iii) $a_{pk}=0$ *and* $a_{iq}=0$ *imply* $a_{pq}=0$ $(i<p<j,k<q<l)$, *and*
(iv) $a_{pk}=1$ *and* $a_{iq}=1$ *imply* $a_{pq}=1$ $(i<p<j,k<q<l)$.

Proof. We first show that if one of the conditions (i)–(iv) does not hold, then A does not cover A'. First suppose that $a_{pk}=0$ and $a_{pl}=1$ for some p with $i<p<j$. Then a sequence of two $L_2\to I_2$ interchanges transforms A into A' with the intermediate matrix C satisfying $A'\prec_{\hat{B}}C\prec_{\hat{B}}A$. Hence A does not cover A' in this case. A similar argument works when $a_{pk}=1$ and $a_{pl}=0$, and when $a_{iq}=0$ and $a_{jq}=1$, or $a_{iq}=1$ and $a_{jq}=0$ $(k<q<l)$.

Now assume (i) and (ii), and suppose that for some q with $k<q<l$, we have $a_{pk}=0$, $a_{iq}=0$ but $a_{pq}=1$. In this case, a sequence of three interchanges replaces A with A' and we get matrices C_1 and C_2 with $A'\prec_{\hat{B}}C_1\prec_{\hat{B}}C_2\prec_{\hat{B}}A$. For example, and showing only the submatrix $A[\{i,i+1,\ldots,j\},\{k,k+1,\ldots,l\}]$,

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & & & & & 0 \\ 0 & 1 & & & & 0 \\ 1 & & & & & 1 \\ 1 & & & & & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & & & & & 0 \\ 1 & 0 & & & & 0 \\ 1 & & & & & 1 \\ 1 & & & & & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & & & & & 0 \\ 0 & 0 & & & & 1 \\ 1 & & & & & 1 \\ 1 & & & & & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & & & & & 0 \\ 0 & 1 & & & & 0 \\ 1 & & & & & 1 \\ 1 & & & & & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

where unspecified entries do not change. Hence A does not cover A' in this case. A similar argument works when $a_{pk} = 1$, $a_{iq} = 1$, and $a_{pq} = 0$ $(i < p < j, k < q < l)$.

Conversely, suppose that (i)–(iv) hold. Since A and A' agree outside of their contiguous submatrices $A[\{i, i+1, \ldots, j\}, \{k, k+1, \ldots, l\}]$ and $A'[\{i, i+1, \ldots, j\}, \{k, k+1, \ldots, l\}]$, and since the row and column sum vectors of $A[\{i, i+1, \ldots, j\}, \{k, k+1, \ldots, l\}]$, and $A'[\{i, i+1, \ldots, j\}, \{k, k+1, \ldots, l\}]$ are the same, it follows that $\sigma_{uv}(A) = \sigma_{uv}(A')$ for all u, v for which one of the following holds: (1) $u < i$, (2) $v < k$, (3) $u \geqslant j$, and (4) $v \geqslant l$.

We now claim that if there is a matrix C with $A' \prec_{\hat{B}} C \prec_{\hat{B}} A$, then every sequence of $L_2 \to I_2$ interchanges that transforms A into C takes place entirely within $A[\{i, i+1, \ldots, j\}, \{k, k+1, \ldots, l\}]$. Assume that a sequence of $L_2 \to I_2$ interchanges transforms A into C where at least one of these interchanges involves an entry of A outside of $A[\{i, i+1, \ldots, j\}, \{k, k+1, \ldots, l\}]$. First assume that one of these $L_2 \to I_2$ interchanges involves an entry a_{pq} where either $p < i$ or $q > l$. Choose such an entry with p minimal, and with q maximal for this p. Since A and A' agree outside of their submatrices determined by rows $i, i+1, \ldots, j$ and columns $k, k+1, \ldots, l$, either $a_{pq} = a'_{pq} = 0$ or $a_{pq} = a'_{pq} = 1$. First suppose that $a_{pq} = a'_{pq} = 0$. Then in our sequence of $L_2 \to I_2$ interchanges, $a_{pq} = 0$ changes to a 1 and then back to a 0. But for our choice of p and q, the first change is impossible in $L_2 \to I_2$ interchanges. Now suppose that $a_{pq} = a'_{pq} = 1$. Then in our sequence of $L_2 \to I_2$ interchanges, $a_{pq} = 1$ changes to a 0 and then back to a 1. This second interchange is impossible for our choice of p and q. A similar contradiction results if $p > j$ or $q < k$, and our claim holds.

The conditions (i)–(iv) imply that with row and column permutations the matrix $A[\{i, i+1, \ldots, j\}, \{k, k+1, \ldots, l\}]$ can be brought to the form

$$\begin{bmatrix} J & J & X \\ J & L_2 & O \\ Y & O & O \end{bmatrix}, \tag{3}$$

where J designates a matrix of all 1's and where $A[\{i, j\}, \{k, l\}]$ is transformed into the displayed matrix L_2 by these row and column operations. Row and column permutations can change an $L_2 \to I_2$ interchange into a $I_2 \to L_2$ interchange, or vice-versa, but otherwise do not affect the existence of interchanges in submatrices of order 2. In a matrix of the form (3) the only possible interchanges can occur wholly in X, L_2, or Y, and interchanges take a matrix of the form (3) to a matrix of the same form except for the possibility that the L_2 gets replaced with I_2. This implies that the only $L_2 \to I_2$ interchange in $A[\{i, i+1, \ldots, j\}, \{k, k+1, \ldots, l\}]$ that can change any of its corner entries is the $L_2 \to I_2$ interchange involving all the corner entries. It then follows that the only sequence of $L_2 \to I_2$ interchanges that transforms A to A' is the single $L_2 \to I_2$ interchange involving all the corner entries of $A[\{i, i+1, \dots, j\}, \{k, k+1, \dots, l\}]$. Hence A covers A' in the secondary Bruhat order on $\mathscr{A}(R, S)$. $\square$

The following corollary is an immediate consequence of Theorem 3.1. It asserts that if the $L_2 \to I_2$ interchange in a matrix A takes place in consecutive rows and consecutive columns, then A covers the resulting matrix in the secondary Bruhat order.

Corollary 3.2. *Let $A = [a_{ij}]$ be a matrix in $\mathscr{A}(R, S)$ where $A[\{i, i+1\}, \{k, k+1\}] = L_2$. Let A' be the matrix obtained from A by the $L_2 \to I_2$ interchange that replaces $A[\{i, i+1\}, \{k, k+1\}] = L_2$ with I_2. Then A covers A' in the secondary Bruhat order on $\mathscr{A}(R, S)$.*

In the hypothesis of Corollary 3.2 it is clear that A also covers A' in the Bruhat order, but we do not know a characterization of the cover relation for the Bruhat order on $\mathscr{A}(R, S)$. Since the Bruhat order is a refinement of the secondary Bruhat order, it follows that if A covers A' in the Bruhat order but A does not cover A' in the secondary Bruhat order, then A and A' are incomparable in the secondary Bruhat order. Some remarks on the cover relation for classes $\mathscr{A}(n, k)$ are given in Section 5.

4. The class $\mathscr{A}(n, 2)$

In this section we show that the Bruhat order and secondary Bruhat order coincide on $\mathscr{A}(n, 2)$, as they do on $\mathscr{A}(n, 1)$. In the next section we will see that the two orders do not in general coincide on classes $\mathscr{A}(n, 3)$.

We first prove three lemmas. For integers k and l with $k \leqslant l$, we now write $[k, l]$ to denote the set $\{k, k+1, \dots, l\}$.

Lemma 4.1. *Let A and C be matrices in $\mathscr{A}(R, S)$ with $A \prec_B C$, and let i and j be integers with $\sigma_{ij}(A) > \sigma_{ij}(C)$. Let s and t be integers with (s, t) lexicographically maximal such that*

$$(r, c) \in [i, s-1] \times [j, t-1] \Rightarrow \sigma_{rc}(A) > \sigma_{rc}(C).$$

Then there exists $(i_0, j_0) \in [i+1, s] \times [j+1, t]$ with $a_{i_0 j_0} = 1$.

Proof. By the maximality condition of (s, t), there exists a $k \in [i, s-1]$ and an $l \in [j, t-1]$ such that $\sigma_{kt}(A) = \sigma_{kt}(C)$ and $\sigma_{sl}(A) = \sigma_{sl}(C)$. If $a_{rc} = 0$ for every $(r, c) \in [i+1, s] \times [j+1, t]$, then we calculate that

$$\begin{aligned}\sigma_{st}(A) &= \sigma_{sl}(A) + \sigma_{kt}(A) - \sigma_{kl}(A)\\ &< \sigma_{sl}(A) + \sigma_{kt}(A) - \sigma_{kl}(C)\\ &= \sigma_{sl}(C) + \sigma_{kt}(C) - \sigma_{kl}(C)\\ &\leqslant \sigma_{st}(C),\end{aligned}$$

a contradiction to the hypothesis that $A \prec_B C$. $\square$

If M is a square matrix of order n, then M^{at} denotes the *antitranspose* of M, that is, the matrix obtained from M by flipping about the antidiagonal running from position $(1, n)$ down to position $(n, 1)$.

Lemma 4.2. *Let A and C be matrices in $\mathscr{A}(n,k)$. Then for every $(i,j) \in [1,n-1] \times [1,n-1]$,*

$$\sigma_{ij}(A) \geqslant \sigma_{ij}(C) \Leftrightarrow \sigma_{n-i,n-j}(A^{\mathrm{at}}) \geqslant \sigma_{n-i,n-j}(C^{\mathrm{at}}).$$

Proof. We observe that

$$nk = \sigma_{ij}(A) + (ik - \sigma_{ij}(A)) + (jk - \sigma_{ij}(A)) + \sigma_{n-i,n-j}(A^{\mathrm{at}}),$$

and thus

$$\sigma_{ij}(A) = (ik + jk - nk) + \sigma_{n-i,n-j}(A^{\mathrm{at}}).$$

The same inequality holds with A replaced with C. Therefore $\sigma_{ij}(A) \geqslant \sigma_{ij}(C)$ is equivalent to

$$(ij + jk - nk) + \sigma_{n-i,n-j}(A^{\mathrm{at}}) \geqslant (ij + jk - nk) + \sigma_{n-i,n-j}(C^{\mathrm{at}}),$$

and this is equivalent to $\sigma_{n-i,n-j}(A^{\mathrm{at}}) \geqslant \sigma_{n-i,n-j}(C^{\mathrm{at}})$. $\square$

Lemma 4.3. *Let A and C be matrices in $\mathscr{A}(n,k)$ with $A \prec_B C$, and let i and j be integers with $\sigma_{ij}(A) > \sigma_{ij}(C)$. Let s and t be integers with (s,t) lexicographically minimal such that*

$$(r,c) \in [s+1,i] \times [t+1,j] \Rightarrow \sigma_{rc}(A) > \sigma_{rc}(C).$$

Then there exists $(i_0, j_0) \in [s+1,i] \times [t+1,j]$ with $a_{i_0 j_0} = 1$.

Proof. Let $i' = n - i$ and let $j' = n - j$. Since $\sigma_{ij}(A) > \sigma_{ij}(C)$, by Lemma 4.2 we have $\sigma_{i'j'}(A^{\mathrm{at}}) > \sigma_{i'j'}(C^{\mathrm{at}})$. Let $s' = n - s$ and $t' = n - t$. Let $(r,c) \in [i', s'-1] \times [j', t'-1]$. Then $n - r \in [n - (s'-1), n - i'] = [s+1, i]$ and $n - c \in [n - (t'-1), n - j'] = [t+1, j]$, and thus $\sigma_{n-r,n-c}(A) > \sigma_{n-r,n-c}(C)$. By Lemma 4.2, this implies that $\sigma_{rc}(A^{\mathrm{at}}) > \sigma_{rc}(C^{\mathrm{at}})$. We conclude that

$$(r,c) \in [i', s'-1] \times [j', t'-1] \Rightarrow \sigma_{rc}(A^{\mathrm{at}}) > \sigma_{rc}(B^{\mathrm{at}}). \tag{4}$$

Finally, by the minimality condition of (s,t), there exists a $k \in [s,i]$ and an $l \in [t,j]$ such that $\sigma_{kt}(A) = \sigma_{kt}(C)$ and $\sigma_{sl}(A) = \sigma_{sl}(C)$. Let $k' = n - k$ and $l' = n - l$. By Lemma 4.2, we have that $\sigma_{k't'}(A^{\mathrm{at}}) = \sigma_{k't'}(C^{\mathrm{at}})$ and $\sigma_{s'l'}(A^{\mathrm{at}}) = \sigma_{s'l'}(C^{\mathrm{at}})$. This implies that (s', t') is lexicographically maximal with property (4).

We may now apply Lemma 4.1 to A^{at} and C^{at} to conclude that there exists $(i'_0, j'_0) \in [i'+1, s'] \times [j'+1, t']$ such that the (i'_0, j'_0)-entry of A^{at} is 1. Let $i_0 = n - i'_0 + 1$ and $j_0 = n - j'_0 + 1$. Then the (i_0, j_0)-entry of A is 1.

Further, since $i' + 1 \leqslant i'_0 \leqslant s'$,

$$i = n - (i'+1) + 1 \geqslant n - i'_0 + 1 = i_0 \geqslant n - s' + 1 = s + 1,$$

so that $i_0 \in [s+1, i]$. Similarly, since $j' + 1 \leqslant j'_0 \leqslant t'$,

$$j = n - (j'+1) + 1 \geqslant n - j'_0 + 1 = j_0 \geqslant n - t' + 1 = t + 1,$$

and hence $j_0 \in [t+1, j]$. $\square$

Theorem 4.4. *Let A and C be two matrices in $\mathscr{A}(n,2)$. Then $A \prec_B C$ if and only if $A \prec_{\hat{B}} C$. In other words, the Bruhat order and secondary Bruhat order are the same on $\mathscr{A}(n,2)$.*

Proof. We know that $A \prec_{\hat{B}} C$ implies that $A \prec_B C$. So we need only prove that if $A \prec_B C$ then $A \prec_{\hat{B}} C$. It suffices to show this under the assumption that C covers A. So assume that C covers A in the Bruhat order.

Since $A \prec_B C$, there is a position (i, j) such that $a_{ij} = 1$ while $\sigma_{ij}(A) > \sigma_{ij}(C)$ (the lexicographically first position in which Σ_A and Σ_C differ, for example, is easily seen to be such a position). We choose such a position (i, j) with $i + j$ as large as possible.

Applying Lemma 4.1, we choose $(i_0, j_0) \in [i + 1, n] \times [j + 1, n]$ such that $a_{i_0 j_0} = 1$ and for any $(r, c) \in [i, i_0 - 1] \times [j, j_0 - 1]$, $\sigma_{rc}(A) > \sigma_{rc}(C)$. We consider three cases.

Case 1: $a_{i_0 j} = a_{i j_0} = 0$.

In this case, an $I_2 \to L_2$ interchange that replaces $A[\{i, i_0\}, \{j, j_0\}] = I_2$ with L_2 results in a matrix D with $A \prec_{\hat{B}} D$. Now, since for $(r, c) \in [1, n] \times [1, n]$

$$\sigma_{rc}(D) = \begin{cases} \sigma_{rc}(A) - 1, & \text{if } (r, c) \in [i, i_0 - 1] \times [j, j_0 - 1], \\ \sigma_{rc}(A), & \text{otherwise,} \end{cases}$$

and since $\sigma_{rc}(A) > \sigma_{rc}(C)$ for any $(r, c) \in [i, i_0 - 1] \times [j, j_0 - 1]$, then $A \prec_B C$ implies $D \preceq_B C$. Since C covers A in the Bruhat order, we conclude that in fact $D = C$ and hence $A \prec_{\hat{B}} C$.

Case 2: $a_{i_0 j} = 1$

Because $i_0 > i$, by the maximality condition on i and j, we know that $\sigma_{i_0 j}(A) = \sigma_{i_0 j}(C)$. We also know that $\sigma_{i_0 - 1, j}(A) > \sigma_{i_0 - 1, j}(C)$.

We claim that $\sigma_{i_0 - 1, j-1}(A) > \sigma_{i_0 - 1, j-1}(C)$. Suppose to the contrary that this is not so, that is, $\sigma_{i_0 - 1, j-1}(A) = \sigma_{i_0 - 1, j-1}(C)$. By calculation we get

$$\begin{aligned} &1 + \sigma_{i_0 - 1, j}(A) + \sigma_{i_0, j-1}(A) - \sigma_{i_0 - 1, j-1}(A) \\ &\quad = \sigma_{i_0 j}(A) \\ &\quad = \sigma_{i_0 j}(C) \\ &\quad = c_{i_0 j} + \sigma_{i_0 - 1, j}(C) + \sigma_{i_0, j-1}(C) - \sigma_{i_0 - 1, j-1}(C) \\ &\quad \leqslant 1 + \sigma_{i_0 - 1, j}(C) + \sigma_{i_0, j-1}(C) - \sigma_{i_0 - 1, j-1}(C), \end{aligned}$$

and this implies that

$$\sigma_{i_0 - 1, j}(A) + \sigma_{i_0, j-1}(A) \leqslant \sigma_{i_0 - 1, j}(C) + \sigma_{i_0, j-1}(C),$$

and so

$$0 < \sigma_{i_0 - 1, j}(A) - \sigma_{i_0 - 1, j}(C) \leqslant \sigma_{i_0, j-1}(C) - \sigma_{i_0, j-1}(A).$$

Thus

$$\sigma_{i_0, j-1}(C) > \sigma_{i_0, j-1}(A),$$

a contradiction.

Therefore, $\sigma_{i_0 - 1, j-1}(A) > \sigma_{i_0 - 1, j-1}(C)$, and we can apply Lemma 4.3 to obtain $(i_1, j_1) \in [1, i_0 - 1] \times [1, j - 1]$ such that $a_{i_1 j_1} = 1$ and for any $(r, c) \in [i_1, i_0 - 1] \times [j_1, j - 1]$, $\sigma_{rc}(A) > \sigma_{rc}(C)$.

We now consider two subcases.

Subcase (a) $i_1 = i$

In this case we have already identified two positions, namely (i, j) and (i_1, j_1), in row $i_1 = i$ that are occupied by 1's in A. Since any row of A contains exactly two 1's, it follows that $a_{i j_0} = 0$.

Similarly, since we have identified 1's in A at positions (i_0, j) and (i_0, j_0), it follows that $a_{i_0 j_1} = 0$. Finally, note that, in addition,

$$(r, c) \in [i_1, i_0 - 1] \times [j_1, j_0 - 1] \Rightarrow \sigma_{rc}(A) > \sigma_{rc}(C).$$

As in Case 1, an $I_2 \to L_2$ interchange results in a matrix D such that $A \prec_B D \preceq_B C$, implying $C = D$ and hence $A \prec_{\hat{B}} C$.

Subcase (b) $i_1 \neq i$

In this case as well, we have identified two positions, namely (i, j) and (i_0, j), in column j that are occupied by 1's in A. Since any column of A contains exactly two 1's, it follows that $a_{i_1 j} = 0$. Similarly, since we have identified 1's in A at positions (i_0, j) and (i_0, j_0), it follows that $a_{i_0 j_1} = 0$. Finally, note that, in addition,

$$(r, c) \in [i_1, i_0 - 1] \times [j_1, j - 1] \Rightarrow \sigma_{rc}(A) > \sigma_{rc}(C).$$

As above an $I_2 \to L_2$ interchange results in a matrix D that must equal C, and we conclude that $A \prec_{\hat{B}} C$.

Case 3: $a_{ij_0} = 1$

The proof here is completely symmetric to the proof in Case 2 (or we could apply Case 2 to the transpose matrices). □

As we will see in the next section, the conclusion of Theorem 4.4 does not in general hold for the classes $\mathscr{A}(n, 3)$.

5. The classes $\mathscr{A}(n, k)$

In Section 2 we gave an example of two matrices in the same class $\mathscr{A}(R, S)$ each of which is minimal in the secondary Bruhat order but one of which is below the other in the Bruhat order, disproving, in particular, that the two orders are the same on each class $\mathscr{A}(R, S)$. In this section we study the regular classes $\mathscr{A}(n, k)$ in more detail and show, in particular, that even for these classes with $k = 3$, the Bruhat order and secondary Bruhat order can be different.

Consider the three matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

and

$$D = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

By calculation, we have

$$\Sigma_A = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 7 & 8 & 9 \\ 3 & 4 & 5 & 8 & 10 & 12 \\ 3 & 5 & 7 & 10 & 13 & 15 \\ 3 & 6 & 9 & 12 & 15 & 18 \end{bmatrix}, \quad \Sigma_C = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 4 & 5 & 6 \\ 2 & 3 & 4 & 7 & 8 & 9 \\ 3 & 4 & 5 & 8 & 10 & 12 \\ 3 & 5 & 7 & 10 & 13 & 15 \\ 3 & 6 & 9 & 12 & 15 & 18 \end{bmatrix},$$

and

$$\Sigma_D = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 3 \\ 1 & 2 & 2 & 4 & 5 & 6 \\ 2 & 3 & 4 & 7 & 8 & 9 \\ 3 & 4 & 5 & 8 & 10 & 12 \\ 3 & 5 & 7 & 10 & 13 & 15 \\ 3 & 6 & 9 & 12 & 15 & 18 \end{bmatrix}.$$

So we see that

$$\Sigma_A > \Sigma_D > \Sigma_C.$$

Hence,

$$A \prec_B D \prec_B C.$$

In fact, it follows from Theorem 3.1 that C covers both D and A in the secondary Bruhat order. This implies that D and A are incomparable in the secondary Bruhat order. We conclude that *the Bruhat order and secondary Bruhat order are different on* $\mathscr{A}(6, 3)$. (By a case-by-case analysis, one can show that, in fact, D covers A in the Bruhat order.)

The class $\mathscr{A}(n, 1)$, in which the Bruhat order and secondary Bruhat order coincide, has a unique minimal element, namely the identity matrix I_n. Also in $\mathscr{A}(n, 1)$ the permutation matrix D_n with 1's in positions $(1, n), (2, n-1), \ldots, (n, 1)$ is the unique maximal element. From this last observation, it follows that $\mathscr{A}(n, n-1)$ also has a unique minimal element, namely $J_n - D_n$. Of course, the classes $\mathscr{A}(n, 0)$ and $\mathscr{A}(n, n)$ each contain only one matrix and so trivially have a unique minimal matrix. In the next theorem we show that there is only one other family of classes $\mathscr{A}(n, k)$ with a unique minimal element.

Theorem 5.1. *Let n and k be integers with $0 \leqslant k \leqslant n$. Then the class $\mathscr{A}(n, k)$ has a unique minimal element in the secondary Bruhat order, and so in the Bruhat order, if and only if $k = 0, 1, n-1,$ or n, or $n = 2k$. The unique minimal matrix in $\mathscr{A}(2k, k)$ is $J_k \oplus J_k$.*

Proof. In this proof, unless otherwise specified, by minimal we mean minimal in the secondary Bruhat order. As already remarked, $\mathscr{A}(n, k)$ has a unique minimal element if $k = 0, 1, n-1,$ or n. First consider the case in which $n = qk$ with $q \geqslant 3$. The matrix $J_k \oplus \cdots \oplus J_k$ (q J_k's) is a minimal matrix in $\mathscr{A}(qk, k)$. Let J_k^{ul} be the matrix obtained from J_k by replacing the 1 in the upper left corner with a 0, and let J_k^{lr} be the matrix obtained from J_k by replacing the 1 in the lower right corner with a 0. Finally, let $J_k^{\mathrm{ur,ll}}$ be the matrix obtained from J_k by replacing both the 1 in the upper right corner and the 1 in the lower left corner with 0's. For matrices A and B of orders m and n, respectively, let $A \mathbin{\widehat{\oplus}} B$ be the matrix obtained from the direct sum $A \oplus B$ by replacing the 0's in positions $(m, m+1)$ and $(m+1, m)$ with 1's. The operation $\widehat{\oplus}$ is associative. It is now easily verified that the matrix

$$J_k^{\mathrm{lr}} \widehat{\oplus} \underbrace{J_k^{\mathrm{ur,ll}} \widehat{\oplus} \cdots \widehat{\oplus} J_k^{\mathrm{ur,ll}}}_{q-2} \widehat{\oplus} J_k^{\mathrm{ul}}$$

is a minimal matrix in $\mathscr{A}(qk, k)$ different from $J_k \oplus \cdots \oplus J_k$. Thus $\mathscr{A}(qk, k)$ does not have a unique minimal matrix.

We next assume that $k \neq 0, 1, n-1, n$ and that $n \neq qk$ for any $q \geqslant 1$. We refer to the recursive algorithm in [3] for minimal matrices in the Bruhat order (and so in the secondary Bruhat order) and exhibit two different minimal matrices in $\mathscr{A}(n, k)$. Let $n = qk + r$ where $q \geqslant 1$ and $1 \leqslant r \leqslant k-1$. If $q \geqslant 2$, then the algorithm produces the matrix $J_k \oplus \cdots \oplus J_k \oplus X$ where there are $q - 1 J_k$'s and X is the minimal matrix in $\mathscr{A}(k+r, k)$ produced by the algorithm. But it is easy to see that $X \oplus J_k \oplus \cdots \oplus J_k$ is also a minimal matrix, different from the first. Now assume that $q = 1$. Then the algorithm produces the matrix

$$M = \left[\begin{array}{c|c} J_{r,k} & O_k \\ \hline X & J_{k,r} \end{array}\right],$$

where X is the minimal matrix in $\mathscr{A}(k, k-r)$ produced by the algorithm. The matrix M is not symmetric, and its transpose M^{t} is a minimal matrix in $\mathscr{A}(n, k)$ different from M.

To complete the proof, we need to show that $J_k \oplus J_k$ is the unique minimal matrix in $\mathscr{A}(2k, k)$, $(k \geqslant 2)$. Let

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_4 & A_3 \end{array}\right],$$

where A_1, A_2, A_4, A_3 are matrices of order k. If $A_1 = J_k$, then $A_3 = J_k$ and $A_2, A_4 = O_k$, and hence $A = J_k \oplus J_k$. Suppose that $A_1 \neq J_k$ so that $A_3 \neq J_k$ and $A_2, A_4 \neq O_k$. For each i, the number of 0's in row i of A_1 (respectively, A_3) equals the number of 1's in row i of A_2 (respectively, A_4). Similarly, for each j, the number of 0's in column j of A_1 (respectively, A_3) equals the number of 1's in column j of A_4 (respectively, A_2). We show by induction that A is not a minimal matrix in $\mathscr{A}(2k, k)$.

Consider the directed bipartite graph Γ with vertex set $X_1 \cup X_2 \cup Y_1 \cup Y_2$, where X_1 and X_2 correspond, respectively, to rows $1, 2, \ldots, k$ and rows $k+1, k+2, \ldots, 2k$ of A, and Y_1 and Y_2 correspond, respectively, to columns $1, 2, \ldots, k$ and columns $k+1, k+2, \ldots, 2k$ of A. The arcs of Γ from X_1 to Y_1 correspond to the 0's of A_1, and the arcs from X_2 to Y_2 correspond to the 0's of A_3. Similarly, the arcs of Γ from Y_1 to X_2 correspond to the 1's of A_4, and the arcs from Y_2 to X_1 correspond to the 1's of A_4. It follows from the comments above that for each vertex u of Γ, the indegree of u equals the outdegree of u. Hence Γ has a directed cycle γ, and the definition of Γ implies that γ alternates cyclically from X_1 to Y_1 to X_2 to Y_2, and then back to X_1 again. The length of γ is $4t$ for some positive integer t. If $t = 1$, then γ corresponds to a submatrix of A equal to L_2. Hence A is not a minimal matrix. Suppose that $t \geqslant 2$. Consider the arc of γ from a vertex in X_1 to a vertex in Y_1 that corresponds to a 0 in position (p, q) of A_1 with the smallest row index p, and the arcs of Γ from Y_1 to X_2 (corresponding to a 1 in some position (r, q) of A_4) and from Y_2 to X_1 (corresponding to a 1 in some position (p, s) of A_2) that immediately follow it, and respectively, immediately precede it. If there is a 0 in position (r, s) of A_4, then A contains a submatrix equal to L_2 and A is not a minimal matrix. If there is a 1 in position (r, s) of A_4, then we see that γ can be replaced with a directed cycle of length $4t'$ for some $t' < t$. It thus follows by induction on t, that A is not a minimal matrix, and thus that $J_k \oplus J_k$ is the unique minimal matrix in $\mathscr{A}(2k, k)$. $\square$

6. Concluding remarks

The algorithm in [3] for producing a minimal matrix in a class $\mathscr{A}(n,k)$ can, in general, be carried out in many ways and so leads to different matrices in $\mathscr{A}(n,k)$ which are minimal in the Bruhat order. But not every minimal matrix results by application of the algorithm. For example, the following matrices in $\mathscr{A}(7,4)$ are minimal:

$$A_1 = \begin{bmatrix} J_{3,4} & O_3 \\ I_4 & J_{4,3} \end{bmatrix}, \quad A_2 = \begin{bmatrix} J_{4,3} & I_4 \\ O_3 & J_{3,4} \end{bmatrix} = A_1^{\mathrm{t}},$$

and

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The matrices A_1 and A_2 arise from the algorithm (by applying it in different ways); the matrix A_3 does not result from the algorithm. A computer search has revealed that A_1, A_2, and A_3 are the only minimal matrices in $\mathscr{A}(7,4)$.

The classical Bruhat order on the set S_n of permutations of order n is a graded order; permutations are graded by the number of their inversions. Neither the Bruhat nor the secondary Bruhat order on $\mathscr{A}(R,S)$ is, in general, graded. To see this we consider $\mathscr{A}(4,2)$ on which the two orders coincide. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Also let

$$X_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad Y_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

By Theorem 3.1 (in fact, by Corollary 3.2 in all but one case), A covers X_1, X_1 covers C, A covers Y_1, Y_1 covers Y_2, Y_2 covers Y_3, and Y_3 covers C. Thus there are maximal chains from A to C of lengths 2 and 4 in the Bruhat order, implying that the Bruhat order is not graded on $\mathscr{A}(4,2)$.

We conclude with some questions that we have not yet considered in much detail.

1. Although the Bruhat order and secondary Bruhat order do not coincide in general on $\mathscr{A}(n,k)$, $k \geqslant 3$, it is possible that the minimal elements in the two orders are the same. Is a matrix A in $\mathscr{A}(n,k)$ minimal in the Bruhat order if and only if A contains no submatrix equal to L_2?

2. We have characterized the cover relation of the secondary Bruhat order on classes $\mathscr{A}(R, S)$; see Theorem 3.1. Is there a nice characterization of the cover relation of the Bruhat order on $\mathscr{A}(R, S)$?
3. Investigate the following partial order on $\mathscr{A}(R, S)$: $A_1 \preceq_{B'} A_2$ if and only if A_2 can be transformed into A_1 by a sequence of $L_2 \to I_2$ interchanges where in each such interchange, L_2 is a submatrix formed by two consecutive rows and two consecutive columns (or, two consecutive rows and any two columns).
4. In classes $\mathscr{A}(2k, k)$, is the maximal length of a chain in the Bruhat order from the minimal element $J_k \oplus J_k$ to the maximal element

   $$\begin{bmatrix} O_k & J_k \\ J_k & O_k \end{bmatrix}$$

   equal to $4k^2$? (This is most likely true and may not be difficult to prove.)
5. What is the largest size of an antichain in the Bruhat order on classes $\mathscr{A}(2k, k)$? More generally, on $\mathscr{A}(R, S)$?
6. The definition of the Bruhat order on $\mathscr{A}(R, S)$ provides an efficient method to check whether $A_1 \preceq A_2$ via the matrices Σ_{A_1} and Σ_{A_2}. Is there an efficient way to check whether $A_1 \preceq_{\hat{B}} A_2$?

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